Euler's Gem—The Polyhedron Formula and the Birth of Topology

by David S. Richeson

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REVIEWED BY JEANINE DAEMS

"They all missed it." Richeson's book begins with a strong and clear motivation for one of his key points on the nature and the historical development of mathematics. "It" is "Euler's Gem," Euler's polyhedron formula, one of the most beautiful formulas of mathematics (in fact, the author informs us, a survey of mathematicians found its beauty to be second only to $e^{\pi i} + 1 = 0$, also Euler's). "They" refers to all of Euler's predecessors who, though active in the field of geometry, failed to come across this elegant and, to our eyes, even obvious relationship.

Euler's polyhedron formula is elegant and simple: In a polyhedron, the number of vertices (V), edges (E) and faces (F) always satisfy the equality $V - E + F = 2$. For example, a cube contains 8 vertices, 12 edges and 6 faces, and indeed, $8 - 12 + 6 = 2$.

But if this formula is so simple, why did no one think of it earlier, especially when, as Richeson explains, people had been fascinated by polyhedra for millennia? The ancient Greeks, for example, were already able to prove that there are exactly five regular polyhedra. Polyhedra are very familiar mathematical objects: They are three-dimensional objects constructed from polygon faces, such as the cube, pyramids, the soccer-ball-shaped truncated icosahedron, and so on. However, there is no historical consensus about the precise definition of a polyhedron. The Greeks and Euler, for example, implicitly assumed that polyhedra are convex, whereas modern definitions do not. And is a polyhedron solid, or is it hollow?

Richeson uses Euler's polyhedron formula as a guiding line on his enthusiastic tour of the wonderful world of geometry and topology. The first part of the book deals with the history of the polyhedron formula, starting with a biographical chapter on Euler. Then Richeson discusses the five regular polyhedra, Pythagoras and Plato, Euclid's "Elements," Kepler's polyhedral universe, and of course Euler's discovery of his polyhedron formula. And he explains why Euler's treatment was new: Until then, the theory of polyhedra had dealt with metric properties of polyhedra like measuring angles, finding lengths of sides and areas of faces, and so on. Euler, however, tried to classify polyhedra by *counting* their features. He was the first one to recognize that "edge" is a useful concept, and he realized it was the vertices, edges and faces he had to count. However, Euler's proof of his formula did overlook some subtleties and is not completely rigorous by modern standards.

Then there is an interesting chapter on Descartes (1596–1650). In 1860 some long lost notes of Descartes surfaced in which he stated a theorem that looks a lot like Euler's polyhedron formula: $P = 2F + 2V - 4$, where P is the number of planar angles in a polyhedron, V the number of vertices and F the number of faces. Since the number of planar angles in a polyhedron is twice the number of edges, Euler's formula follows easily (*if* one knows the concept of an edge, and it was Euler who introduced that). So, whether Descartes did or did not prediscover Euler's formula is debatable, but Richeson decides it is not unreasonable to continue ascribing it to Euler.

Legendre (1752–1833) gave a proof of Euler's formula that is correct by our standards, using a projection of the polyhedron on a sphere. A little later it was noticed that Legendre's proof even worked for a bigger class of polyhedra than the convex ones: The so-called star-convex polyhedra.

After this historical exposition, Richeson proceeds by discussing some aspects of more modern mathematics that all have something to do with the polyhedron formula. This part of the book contains some elements of graph theory, the four-color theorem, the discussion of which kinds of polyhedra are exceptions to Euler's formula and generalizations of the formula that arose from this, and eventually the rise of topology.

Does Euler's formula also apply to objects other than polyhedra? Yes. For example, it applies to partitions of the sphere, something Legendre already used in his proof. Cayley noticed that when Euler's formula is applied to graphs, the edges need not be straight. Richeson uses such ideas to illustrate the transition from a geometric to a topological way of thinking about shapes. He explains very clearly that in geometry it is crucial that the objects are rigid, but sometimes these rigid features of geometric objects obscure the underlying structures.

Richeson's introduction to topology is very nice. He explains what surfaces are, describes objects like the Möbius strip, the Klein bottle and the projective plane, discusses when objects are topologically the same, states a theorem that relates Euler's formula to surfaces, gives an introduction to knot theory, differential equations, the hairy ball theorem, the Poincaré conjecture… The book treats too many subjects to mention all of them. They are all related to Euler's polyhedron formula in some sense, and together they give a very good overview of the field of topology and its history.

But that is not all Richeson achieves with this book: He also shows what it is that mathematicians *do*. He shows that mathematics is created by people and that it changes over time. Usually, theorems were not stated originally in their current formulation.

Richeson's book is definitely not a mathematical textbook, and it is not just a historical story either. He wants to show what he enjoys about the topology he works on as a research mathematician. As he writes in the preface: "It is my experience that the general public has little idea what mathematics is and certainly has no conception what a research mathematician studies. They are shocked to discover that new mathematics is [sic] still being created." And he tells us why he was attracted to topology: "The loose and flexible topological

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view of the world felt very comfortable. Geometry seemed straight-laced and conservative in comparison. If geometry is dressed in a suit coat, topology dons jeans and a T-shirt."

His playful attitude to mathematics is clearly expressed in the book: There is an abundance of examples, and there are even templates for building your own platonic solids, as well as the Möbius band, the Klein bottle and the projective plane.

As he mentions in the preface, Richeson wrote his book for both a general audience and for mathematicians. I think he succeeded. Many insights and theorems he explains are difficult and quite deep. He skips the formal details but does not leave out the mathematical reasoning. And he keeps a good balance between the mathematical arguments and intuitive insights.

His explanations are appealing. An example is: "Even more bizarre, could it [the universe] be nonorientable? Is it possible for a right-handed astronaut to fly away from earth, and return left-handed?" The focus of the book lies on the big picture, and for the interested reader there is a list of recommended reads, as well as a long list of references containing many primary sources for the historical part.

The fact that "Euler's gem" has no formal prerequisites does not make it an easy book. As Richeson writes in the preface: "Do not be misled, though—some of the ideas are quite sophisticated, abstract and challenging to visualize. … Reading mathematics is not like reading a novel." Which is true. But Richeson believes the audience for this book is self-selecting: "Anyone who *wants* to read it should be *able* to read it."

I liked Richeson's style of writing. He is enthusiastic and humorous. It was a pleasure reading this book, and I recommend it to everyone who is not afraid of mathematical arguments and has ever wondered what this field of "rubber-sheet geometry" is about. You will not be disappointed.

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